Working Your Quads

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THE QUADRATIC FORMULA is the Rodney Dangerfield of algebra. Even though it's one of the all-time greats, it don't get no respect.

Professionals certainly aren't enamored of it. When mathematicians and physicists are asked to list the top ten most beautiful or important equations of all time, the quadratic formula never makes the cut. Oh sure, everybody swoons over $1 + 1 = 2$, and $E = mc^2$, and the pert little Pythagorean theorem, strutting like it's all that just because $a^2 + b^2 = c^2$. But the quadratic formula? Not a chance.

Admittedly, it's unsightly. Some students prefer to sound it out, treating it as a ritual incantation: "*x* equals negative *b*, plus or minus the square root of *b* squared minus four *a c*, all over two *a*." Others made of sterner stuff look the formula straight in the face, confronting a hodgepodge of letters and symbols more formidable than anything they've encountered up to that point:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It's only when you understand what the quadratic formula is trying to do that you can begin to appreciate its inner beauty. In this chapter I hope to give you a feeling for the cleverness packed into that porcupine of symbols, along with a better sense of what the formula means and where it arises.

There are many situations in which we'd like to figure out the value of some unknown number. What dose of radiation therapy should be given to shrink a thyroid tumor? How much money would you have to pay each month to cover a thirty-year mortgage of $200,000 at a fixed annual interest rate of 5 percent? How fast does a rocket have to go to escape the Earth's gravity?

Algebra is the place where we cut our teeth on the simplest problems of this type. The subject was developed by Islamic mathematicians around A.D. 800, building on earlier work by Egyptian, Babylonian, Greek, and Indian scholars. One practical impetus at that time was the challenge of calculating inheritances according to Islamic law.

For example, suppose a widower dies and leaves his entire estate of 10 dirhams to his daughter and two sons. Islamic law requires that both the sons must receive equal shares. Moreover, each son must receive twice as much as the daughter. How many dirhams will each heir receive?

Let's use the letter x to denote the daughter's inheritance. Even though we don't know what x is yet, we can reason about it as if it were an ordinary number. Specifically, we know that each son gets twice as much as the daughter does, so they each receive $2x$. Thus, taken together, the amount that the three heirs inherit is $x + 2x + 2x$, for a total of $5x$, and this must equal the total value of the estate, 10 dirhams. Hence $5x = 10$ dirhams. Finally, by dividing both sides of the equation by 5, we

[The Joy of X by Steven Strogatz]

see that $x = 2$ dirhams is the daughter's share. And since each of the sons inherits $2x$, they both get 4 dirhams.

Notice that two types of numbers appeared in this problem: known numbers, like 2, 5, and 10, and unknown numbers, like x. Once we managed to derive a relationship between the unknown and the known (as encapsulated in the equation $5x = 10$), we were able to chip away at the equation, dividing both sides by 5 to isolate the unknown x. It was a bit like a sculptor working the marble, trying to release the statue from the stone.

A slightly different tactic would have been needed if we had encountered a known number being *subtracted* from an unknown, as in an equation like $x - 2 = 5$. To free x in this case, we would pare away the 2 by adding it to both sides of the equation. This yields an unencumbered x on the left and $5 + 2 = 7$ on the right. Thus $x = 7$, which you may have already realized by common sense.

Although this tactic is now familiar to all students of algebra, they may not realize the entire subject is named after it. In the early part of the ninth century, Muhammad ibn Musa al-Khwarizmi, a mathematician working in Baghdad, wrote a seminal textbook in which he highlighted the usefulness of restoring a quantity being subtracted (like 2, above) by adding it to the other side of an equation. He called this process *al-jabr* (Arabic for "restoring"), which later morphed into "algebra." Then, long after his death, he hit the etymological jackpot again. His own name, al-Khwarizmi, lives on today in the word "algorithm."

In his textbook, before wading into the intricacies of calculating inheritances, al-Khwarizmi considered a more complicated class of equations that embody relationships among

three kinds of numbers, not the mere two considered above. Along with known numbers and an unknown (x), these equations also included the square of the unknown (x^2). They are now called quadratic equations, from the Latin *quadratus*, for "square." Ancient scholars in Babylonia, Egypt, Greece, China, and India had already tackled such brainteasers, which often arose in architectural or geometrical problems involving areas or proportions, and had shown how to solve some of them.

An example discussed by al-Khwarizmi is

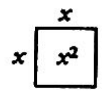
$$x^2 + 10x = 39.$$

In his day, however, such problems were posed in words, not symbols. He asked: "What must be the square which, when increased by ten of its own roots, amounts to thirty-nine?" (Here, the term "root" refers to the unknown x.)

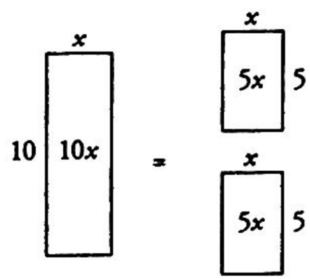
This problem is much tougher than the two we considered above. How can we isolate x now? The tricks used earlier are insufficient, because the x^2 and $10x$ terms tend to step on each other's toes. Even if you manage to isolate x in one of them, the other remains troublesome. For instance, if we divide both sides of the equation by 10, the $10x$ simplifies to x, which is what we want, but then the x^2 becomes $x^2/10$, which brings us no closer to finding x itself. The basic obstacle, in a nutshell, is that we have to do two things at once, and they seem almost incompatible.

The solution that al-Khwarizmi presents is worth delving into in some detail, first because it's so slick, and second because it's so powerful—it allows us to solve *all* quadratic equations in a single stroke. By that I mean that if the known numbers 10 and 39 above were changed to any other numbers, the method would still work.

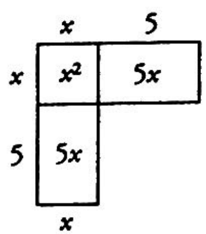
The idea is to interpret each of the terms in the equation geometrically. Think of the first term, x^2, as the area of a square with dimensions x by x.



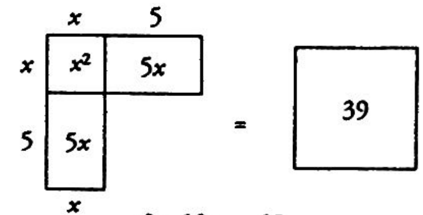
Similarly, regard the second term, $10x$, as the area of a rectangle of dimensions 10 by x or, more ingenious, as the area of two equal rectangles, each measuring 5 by x. (Splitting the rectangle into two pieces sets the stage for the key maneuver that follows, known as completing the square.)



Attach the two new rectangles onto the square to produce a notched shape of area $x^2 + 10x$:

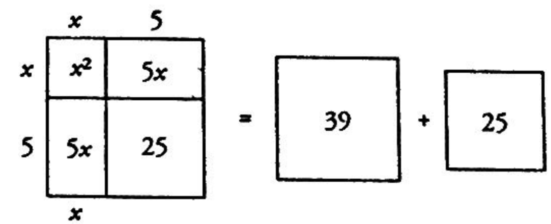


Viewed in this light, al-Khwarizmi's puzzle amounts to asking: If the notched shape occupies 39 square units of area, how large would x have to be?



$$x^2 + 10x = 39$$

The picture itself suggests an almost irresistible next step. Look at that missing corner. If only it were filled in, the notched shape would turn into a perfect square. So let's take the hint and complete the square.



$$(x + 5)^2 = 64$$

Supplying the missing 5×5 square adds 25 square units to the existing area of $x^2 + 10x$, for a total of $x^2 + 10x + 25$. Equivalently, that combined area can be expressed more neatly as $(x + 5)^2$, since the completed square is $x + 5$ units long on each side.

The upshot is that x^2 and $10x$ are now moving gracefully

as a couple, rather than stepping on each other's toes, by being paired within the single expression $(x + 5)^2$. That's what will soon enable us to solve for x.

Meanwhile, because we added 25 units of area to the left side of the equation $x^2 + 10x = 39$, we must also add 25 to the right side, to keep the equation balanced. Since $39 + 25 = 64$, our equation then becomes

$$(x + 5)^2 = 64.$$

But that's a cinch to solve. Taking square roots of both sides gives $x + 5 = 8$, so $x = 3$.

Lo and behold, 3 really does solve the equation $x^2 + 10x = 39$. If we square 3 (giving 9) and then add 10 times 3 (giving 30), the sum is 39, as desired.

There's only one snag. If al-Khwarizmi were taking algebra today, he wouldn't receive full credit for this answer. He fails to mention that a negative number, $x = -13$, also works. Squaring it gives 169; adding it ten times gives -130; and they too add up to 39. But this negative solution was ignored in ancient times, since a square with a side of negative length is geometrically meaningless. Today, algebra is less beholden to geometry and we regard the positive and negative solutions as equally valid.

In the centuries after al-Khwarizmi, scholars came to realize that *all* quadratic equations could be solved in the same way, by completing the square—as long as one was willing to allow the negative numbers (and their bewildering square roots) that often came up in the answers. This line of argument revealed that the solutions to any quadratic equation

$$ax^2 + bx + c = 0$$

(where a, b, and c are known but arbitrary numbers, and x is unknown) could be expressed by the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What's so remarkable about this formula is how brutally explicit and comprehensive it is. There's the answer, right there, no matter what a, b, and c happen to be. Considering that there are infinitely many possible choices for each of them, that's a lot for a single formula to manage.

In our own time, the quadratic formula has become an irreplaceable tool for practical applications. Engineers and scientists use it to analyze the tuning of a radio, the swaying of a footbridge or a skyscraper, the arc of a baseball or a cannonball, the ups and downs of animal populations, and countless other real-world phenomena.

For a formula born of the mathematics of inheritance, that's quite a legacy.